International Journal of Theoretical Physics, Vol. 10, No. 5 (1974), pp. 333–353

A Galactic Model II: The Gravitationally Radiated Angular and Linear Momenta Fluxes

DEXTER J. BOOTH

Department of Mathematics and Computer Studies, the Polytechnic, Queensgate, Huddersfield HD1 3DH

Received: 26 October 1973

Abstract

Using the linearised theory of general relativity the gravitationally radiated angular and linear momenta from a galactic model of N gravitational radiators is calculated. The results are presented in terms of the lowest order contributing multipole moments (quadrupole), the orientations of the radiators about a common reference frame, the distances between pairs of radiators and the frequency of each radiator. This work is a continuation of an earlier work in which the galactic model was first proposed and its gravitationally radiated energy flux was computed.

1. *Introduction*

In an earlier paper by the present author (Booth, 1973) a Galactic Model was proposed and its gravitationally radiated power flux was computed in terms of the quadrupole moments of the individual galactic sources and their mutual separations. In this work the model is taken a stage further and the gravitationally radiated angular and linear momenta fluxes are computed—again in terms of the quadrupole moments of the galactic sources and their mutual separations.

The gravitationally radiated angular momentum emitted by a pair of sources has been previously found by Booth, Cooperstock & Rumsey (1972) but, as in the case of the previous work on power emission, the restricted orientation imposed on the separation of the sources prevents a galactic model being constructed from arbitrarily orientated sources and so the problem is here reworked without restriction on source orientation.

Whereas a single isolated source will only radiate linear momentum flux, at the lowest multipole order, through the quadrupole-octupole mode (Bonnor & Rotenberg, 1961, 1965; Papapetrou, 1962; Peres, 1962) it has been shown by Cooperstock & Booth (1969b) that quadrupole-quadrupole linear momentum flux does exist for pairs of sources as a consequence of their mutual interaction. Again this work was performed under the imposition of restrictive source orientations and, as the energy and angular-momentum formlations had to be reworked, so is the linear momentum formulation.

2. *Angular Momentum Conservation*

In special relativity the energy-momentum tensor T_i^j for a material stress-energy distribution has a vanishing divergence

$$T_{i\ ,j}^{\ j} = 0 \tag{2.1}$$

and consequently readily lends itself to integral conservation laws of energy and momentum. However, in the theory of general relativity the covariant generalisation of equation (2.1) is the vanishing covariant divergence

$$T_{i;j}^{j} \equiv \frac{1}{\sqrt{(-g)}}(\sqrt{(-g)}T_i^j)_{,j} - g_{lm,i}T^{lm} = 0 \tag{2.2}$$

which does not readily lend itself to integral conservation laws. The reason being that there is now a gravitational field contribution to the energy and momentum which is not contained in the energy-momentum tensor T_i^j. To include the gravitational field as well as the stress-energy distribution in generalised concepts of energy and momentum conservation, pseudotensorial quantities t_i^j are constructed from the field variables in such a manner that these auxiliary quanties, together with the energy-momentum tensor T_i^j, have a vanishing divergence in like manner to equation (2.1). This construction, which can be achieved in an infinite number of ways, was performed by Landau & Lifshitz (1965) who were able to develop an expression for the gravitational field energy-momentum pseudotensor t_i^j which contains first and no higher derivatives of the metric tensor g_{ij} (Møller, 1966) and which, moreover, is symmetric making it possible to define a conserved angular momentum for the material distribution plus the gravitational field in a natural manner.

The Landau-Lifshitz pseudotensor t^{ij} satisfies the equation

$$(-g)(T^{ij} + t^{ij}) = h^{ijl}_{\ \ ,l} \tag{2.3}$$

where†

$$h^{ijl} = \frac{c^4}{16\pi G}\{(-g)(g^{ij}g^{lm} - g^{il}g^{jm})\}_{,m} \tag{2.4}$$

clearly

$$h^{ijl}_{\ \ ,lj} = 0 \tag{2.5}$$

so that the four-momentum for the matter plus the gravitational field

$$P^i = \frac{1}{c}\int (-g)(T^{ij} + t^{ij})\, dS_j \tag{2.6}$$

† The system of coordinates is chosen so that $g_{ij} \to \text{diag}(-1, 1, 1, 1)$ as $r \to \infty$.

satisfies the conservation laws

$$P^i{}_{,i} = 0 \tag{2.7}$$

The generalised angular momentum tensor $\mathscr{L}^{ij}$ is defined in terms of the generalised four-momentum density as

$$\begin{aligned}\mathscr{L}^{ij} &= \int (x^i\, dP^j - x^j\, dP^i)\\ &= \frac{1}{c}\int (x^i h^{jlm}{}_{,m} - x^j h^{ilm}{}_{,m})\, dS_l\end{aligned} \tag{2.8}$$

Angular momentum conservation can be expressed by the condition that the angular momentum density has a vanishing divergence

$$\{x^i h^{jlm}{}_{,m} - x^j h^{ilm}{}_{,m}\}_{,l} = 0 \tag{2.9}$$

Hence, choosing the hypersurface x^0 = constant for the integration in equation (2.8) it is readily shown that the radiated angular momentum flux is given by†

$$\frac{d\mathscr{L}^{ij}}{dt} = -\oint_S (-g)(x^i t^{j\alpha} - x^j t^{i\alpha}) n_\alpha\, dS \tag{2.10}$$

where the surface S bounds the volume containing the material distribution and is chosen sufficiently remote enough to make the material stress-energy tensor vanish on S.

The angular momentum flux emitted by a single isolated source has been found previously to be given by the expression (Morgan & Peres, 1963; Peters, 1969; Cooperstock & Booth, 1969a)

$$\frac{d\mathscr{L}^\delta}{dt} = \frac{\epsilon_{\delta\beta\gamma}}{2}\frac{d\mathscr{L}^{\beta\gamma}}{dt} = \epsilon_{\delta\beta\gamma}\cdot\frac{2G}{5c^4}\ddot{D}^{\gamma\alpha}\dddot{D}^{\beta\alpha} \tag{2.11}$$

where

$$D^{\alpha\beta} = d^{\alpha\beta} - \tfrac{1}{3}\delta^{\alpha\beta} d^{\gamma\gamma} \tag{2.12}$$

is the quadrupole moment of the source and‡ (Papapetrou, 1962; Booth, 1973)

$$d^{\alpha\beta} = \int T^{00}\xi^\alpha \xi^\beta\, d^3\xi \tag{2.13}$$

3. *Interaction Angular Momentum*

Using Einstein's linearised metric the Landau-Lifshitz pseudotensor can be written in the form§ (Cooperstock & Booth, 1969a)

$$(-g)t^{ij} = (-g)s^{ij} + \overset{(0)}{g}{}^{ij}\Phi \tag{3.1}$$

† n_α is the outward drawn normal to S.

‡ x^i is a field variable and ξ^α is a source variable.

§ Where $\overset{(0)}{g}_{ij} = \mathrm{diag}\,(-1, 1, 1, 1)$. $\overset{(0)}{g}{}^{ij}\Phi$ does not contribute to equation (2.10).

where

$$
\begin{aligned}
(-g)s^{ij} = \frac{c^4}{16\pi G}\{&\Psi^{ij}{}_{,l}\Psi^{lm}{}_{,m} - \Psi^{il}{}_{,l}\Psi^{jm}{}_{,m} \\
&-\overset{(0)}{g}{}^{il}\overset{(0)}{g}{}_{mn}\Psi^{jn}{}_{,p}\Psi^{mp}{}_{,l} - \overset{(0)}{g}{}^{jl}\overset{(0)}{g}{}_{mn}\Psi^{in}{}_{,p}\Psi^{mp}{}_{,l} \\
&+\tfrac{1}{4}(2\overset{(0)}{g}{}_{np}\overset{(0)}{g}{}_{qr} - \overset{(0)}{g}{}_{pq}\overset{(0)}{g}{}_{nr})\overset{(0)}{g}{}^{il}\overset{(0)}{g}{}^{jm}\Psi^{nr}{}_{,l}\Psi^{pq}{}_{,m} \\
&+\overset{(0)}{g}{}_{lm}\overset{(0)}{g}{}^{np}\Psi^{il}{}_{,n}\Psi^{jm}{}_{,p}\}
\end{aligned}
\tag{3.2}
$$

and where

$$\Psi_i{}^j = \frac{4G}{c^4}\int T_i{}^j(t - R/c)\frac{dV}{R} \tag{3.3}$$

are retarded potential solutions to Einstein's linearised field equations.

The galactic model consists of a material distribution of N gravitational radiators† (Booth, 1973) and if $\overset{(n)}{\psi}{}_i{}^j$ denotes the field of the nth radiator, the total field is given as

$$\Psi_i{}^j = \sum_{n=1}^{N} \overset{(n)}{\psi}{}_i{}^j \tag{3.4}$$

From equations (3.1), (3.2) and (3.4) it can be seen that equation (2.10) consists of two types of integral; one type containing terms quadratic in the field of a given radiator and the other containing products of fields of pairs of radiators. The sum of all integrals of the former type yields the angular momentum loss of the radiators in the absence of interaction and the latter integrals yield the interaction angular momentum flux between pairs of radiators.

As in the calculation of the galactic power loss (Booth, 1973) the retarded potential field solutions of Einstein's linearised field equations must be expanded about the retarded time $t - \bar{r}/c$ to yield‡

$$\bar{\psi}^{ij} = \frac{4G}{c^4}\int \{\bar{T}^{ij} + (\bar{r} - \bar{R})\bar{T}^{ij}{}_{,0} + \tfrac{1}{2}(\bar{r} - \bar{R})^2\bar{T}^{ij}{}_{,00} + \cdots\}\frac{dV}{\bar{R}} \tag{3.5}$$

where

$$\bar{T}^{ij} \equiv \bar{T}^{ij}(t - \bar{r}/c) \tag{3.6}$$

The geometrical arrangement is illustrated in Fig. 1.

It should at this stage be mentioned that whereas in the galactic power loss calculation the wave functions were only expanded up to order r^{-1}, here they

† The word 'radiator' is used throughout to denote a stress-energy distribution.

‡ A bar over quantities implies a specific radiator, except where otherwise stated, i.e. equation (3.24) et seq.

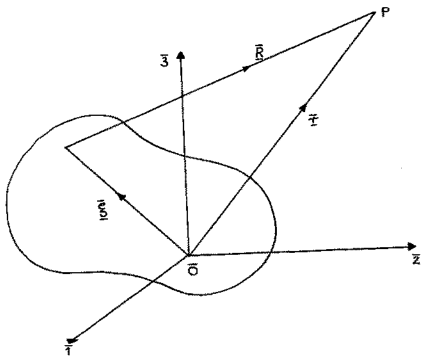

Figure 1.

must be expanded up to order r^{-2} as the numerator of the integrand of equation (2.10) is of order r^3. The contribution to the angular momentum flux will then arise from the terms of order r^{-3} in the wave function products—the terms of order r^{-2} vanishing, as they must, in order that the integral will converge as $S \to \infty$.

Now,

$$\begin{aligned} \bar{R}^2 &= (\bar{x}^\alpha - \bar{\xi}^\alpha)^2 \\ &= \bar{r}^2 - 2\bar{x}^\alpha \bar{\xi}^\alpha + \bar{\xi}^\alpha \bar{\xi}^\alpha \end{aligned} \tag{3.7}$$

where $\bar{\xi}^\alpha$ are spatial source variables and $\bar{R}$ is the source point to field point distance.

Hence,

$$(\bar{r} - \bar{R}) = \bar{n}_\alpha \bar{\xi}^\alpha - \frac{1}{2r}\{\bar{\xi}^\alpha \bar{\xi}^\alpha - (\bar{n}_\alpha \bar{\xi}^\alpha)\} + O(\bar{r}^{-2})$$

$$\frac{1}{\bar{R}} = \frac{1}{\bar{r}} + \frac{\bar{n}_\alpha \bar{\xi}^\alpha}{\bar{r}^2} + O(\bar{r}^{-3}) \tag{3.8}$$

Substitution of equations (3.7) and (3.8) into equation (3.5) yields

$$\begin{aligned} \bar{\psi}^{ij}(t - \bar{r}/c) = {} & \frac{4G}{c^4 \bar{r}} \int \{\bar{T}^{ij} + \bar{n}_\alpha \bar{\xi}^\alpha \bar{T}^{ij}{}_{,0} + \tfrac{1}{2}\bar{n}_\alpha \bar{\xi}^\alpha \bar{T}^{ij}{}_{,00} + \cdots\} \, d^3\bar{\xi} \\ & + \frac{4G}{c^4 \bar{r}^2} \int \{\bar{V}^{ij} + \bar{n}_\alpha \bar{\xi}^\alpha \bar{V}^{ij}{}_{,0} + \tfrac{1}{2}\bar{n}_\alpha \bar{\xi}^\alpha \bar{V}^{ij}{}_{,00} + \cdots\} \, d^3\bar{\xi} \end{aligned} \tag{3.9}$$

where

$$\bar{V}^{ij} \equiv \bar{n}_\alpha \bar{\xi}^\alpha \bar{T}^{ij} - \tfrac{1}{2}(\bar{\xi}^\alpha \bar{\xi}^\alpha - [\bar{n}_\alpha \bar{\xi}^\alpha]^2)\bar{T}^{ij}{}_{,0} \tag{3.10}$$

Equations (3.9) and (3.10) in conjunction with Booth (1973) give the following field components to quadrupole order as,

$$\bar{\psi}^{\alpha\beta}(t-\bar{r}/c)=\frac{2G}{c^4\bar{r}}\,\bar{d}^{\alpha\beta}{}_{,00}$$

$$\bar{\psi}^{\alpha0}(t-\bar{r}/c)=\frac{2G}{c^4\bar{r}}\,\bar{d}^{\alpha\beta}{}_{,00}\bar{n}_\beta+\frac{2G}{c^4\bar{r}^2}\,\bar{d}^{\alpha\beta}{}_{,0}\bar{n}_\beta$$

$$\bar{\psi}^{00}(t-\bar{r}/c)=\frac{2G}{c^4\bar{r}}\,\bar{d}^{\alpha\beta}{}_{,00}\bar{n}_\alpha\bar{n}_\beta+\frac{2G}{c^4\bar{r}^2}\,\{3\bar{d}^{\alpha\beta}{}_{,0}\bar{n}_\alpha\bar{n}_\beta-\bar{d}^{\gamma\gamma}{}_{,0}\} \tag{3.11}$$

where

$$\bar{d}^{\alpha\beta}\equiv\bar{d}^{\alpha\beta}(t-\bar{r}/c) \tag{3.12}$$

Equations (3.11) give a radiator field expanded about a retarded time relative to its own centre of mass and since we have a number of radiators it will eventually be necessary to expand the field of each about a common retarded time $t-r/c$. Meanwhile, from Fig. 2 it is seen that, for asymptotic fields, since

$$\underline{\bar{r}}=\underline{r}-\underline{\bar{L}} \tag{3.13}$$

then

$$\bar{r}^m=r^m\left\{1-\frac{2n_\alpha\bar{L}^\alpha}{r}+\frac{\bar{L}^\alpha\bar{L}^\alpha}{r^2}\right\}^{m/2} \tag{3.14}$$

and

$$\underline{\bar{n}}=\frac{\underline{\bar{r}}}{\bar{r}}=\underline{n}+\frac{1}{r}\,(\underline{n}n_\alpha\bar{L}^\alpha-\underline{\bar{L}})+O(r^{-2}) \tag{3.15}$$

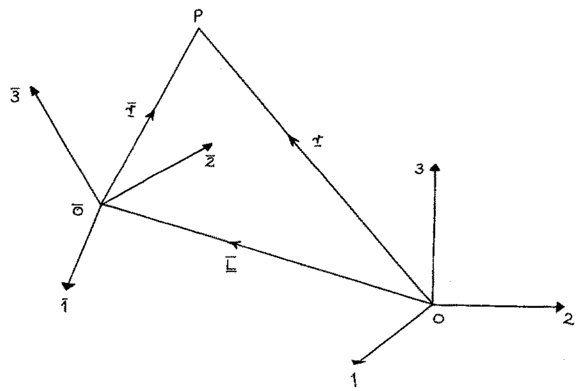

Figure 2.

Using equations (3.13) to (3.15) equations (3.11) can be expanded about the retarded time $t - r/c + n_\alpha \bar{L}^\alpha/c$ to yield†

$$\bar{\psi}^{\alpha\beta} = \frac{2G}{c^4 r}\bar{d}^{\alpha\beta}{}_{,00} + \frac{2G}{c^4 r^2}\left\{\bar{d}^{\alpha\beta}{}_{,00} n_\gamma \bar{L}^\gamma + \frac{\bar{d}^{\alpha\beta}{}_{,000}}{2}\left[n_\gamma n_\delta \bar{L}^\gamma \bar{L}^\delta - \bar{L}^\gamma \bar{L}^\gamma\right]\right\}$$

$$\bar{\psi}^{\alpha 0} = \frac{2G}{c^4 r}\bar{d}^{\alpha\beta}{}_{,00} n_\beta + \frac{2G}{c^4 r^2}\left\{2\bar{d}^{\alpha\beta}{}_{,00} n_\beta n_\gamma \bar{L}^\gamma - \bar{d}^{\alpha\beta}{}_{,00}\bar{L}^\beta + \bar{d}^{\alpha\beta}{}_{,0} n_\beta \right.$$
$$\left. + \frac{\bar{d}^{\alpha\beta}{}_{,000}}{2} n_\beta \left[n_\gamma n_\delta \bar{L}^\gamma \bar{L}^\delta - \bar{L}^\gamma \bar{L}^\gamma\right]\right\}$$

$$\bar{\psi}^{00} = \frac{2G}{c^4 r}\bar{d}^{\alpha\beta}{}_{,00} n_\alpha n_\beta + \frac{2G}{c^4 r^2}\left\{3\bar{d}^{\alpha\beta}{}_{,00} n_\alpha n_\beta n_\gamma \bar{L}^\gamma - \bar{d}^{\gamma\gamma}{}_{,0} \right.$$
$$+ 3\bar{d}^{\alpha\beta}{}_{,0} n_\alpha n_\beta - \bar{d}^{\alpha\beta}{}_{,00}(n_\alpha \bar{L}^\beta + n_\beta \bar{L}^\alpha)$$
$$\left. + \frac{\bar{d}^{\alpha\beta}{}_{,000}}{2} n_\alpha n_\beta \left[n_\gamma n_\delta \bar{L}^\gamma \bar{L}^\delta - \bar{L}^\gamma \bar{L}^\gamma\right]\right\} \tag{3.16}$$

The derivatives of these wave functions (which are listed in Appendix A) are then combined with equations (2.10), (3.1) and (3.2) to yield the time-averaged interaction angular momentum flux between the pth and qth radiators as

$$\langle \dot{\mathscr{L}}^{\alpha\beta}{}_{pq} \rangle = \frac{-G}{4\pi c^4}\int_{4\pi} \left\{\left(\overset{(p)}{\bar{d}}{}^{\beta\phi}{}_{,000} n_\alpha - \overset{(p)}{\bar{d}}{}^{\alpha\phi}{}_{,000} n_\beta\right)\left(3\overset{(q)}{\bar{d}}{}^{\phi\rho}{}_{,00} n_\rho \right.\right.$$
$$- \frac{9}{2}\overset{(q)}{\bar{d}}{}^{\rho\sigma}{}_{,00} n_\phi n_\rho n_\sigma + \tfrac{3}{2}\overset{(q)}{\bar{d}}{}^{\rho\rho}{}_{,00} n_\phi + \left[\overset{(p)}{\bar{L}}{}^\pi - \overset{(q)}{\bar{L}}{}^\pi\right]\left\{\overset{(q)}{\bar{d}}{}^{\phi\pi}{}_{,000}\right.$$
$$\left.\left.\left. + \overset{(q)}{\bar{d}}{}^{\rho\sigma}{}_{,000} n_\phi n_\pi n_\rho n_\sigma + \overset{(q)}{\bar{d}}{}^{\rho\pi}{}_{,000} n_\phi n_\rho - \overset{(q)}{\bar{d}}{}^{\phi\rho}{}_{,000} n_\rho n_\pi\right\}\right)\right\} d\Omega$$
$$- \frac{G}{4\pi c^4}\int_{4\pi} \left\{\left(\overset{(q)}{\bar{d}}{}^{\beta\phi}{}_{,000} n_\alpha - \overset{(q)}{\bar{d}}{}^{\alpha\phi}{}_{,000} n_\beta\right)\left(3\overset{(p)}{\bar{d}}{}^{\phi\rho}{}_{,00} n_\rho \right.\right.$$
$$- \frac{9}{2}\overset{(p)}{\bar{d}}{}^{\rho\sigma}{}_{,00} n_\phi n_\rho n_\sigma + \tfrac{3}{2}\overset{(p)}{\bar{d}}{}^{\rho\rho}{}_{,00} n_\phi + \left[\overset{(q)}{\bar{L}}{}^\pi - \overset{(p)}{\bar{L}}{}^\pi\right]\left\{\overset{(p)}{\bar{d}}{}^{\phi\pi}{}_{,000}\right.$$
$$\left.\left.\left. + \overset{(p)}{\bar{d}}{}^{\rho\sigma}{}_{,000} n_\phi n_\pi n_\rho n_\sigma + \overset{(p)}{\bar{d}}{}^{\rho\pi}{}_{,000} n_\phi n_\rho - \overset{(p)}{\bar{d}}{}^{\phi\rho}{}_{,000} n_\rho n_\pi\right\}\right)\right\} d\Omega$$

† The delay at this stage in not expanding about $t - r/c$ is for computational ease only.

$$
\begin{aligned}
&-\frac{G}{4\pi c^4}\int_{4\pi}\left\{\left(n_\alpha\left[\overset{(p)}{L}{}^\beta+\overset{(q)}{L}{}^\beta\right]-n_\beta\left[\overset{(p)}{L}{}^\alpha+\overset{(q)}{L}{}^\alpha\right]\right)\left(\tfrac{1}{4}\left\{\overset{(p)}{d}{}^{\rho\rho}{}_{,000}\overset{(q)}{d}{}^{\sigma\sigma}{}_{,000}\right.\right.\right.\\
&-2\overset{(p)}{d}{}^{\rho\sigma}{}_{,000}\overset{(q)}{d}{}^{\rho\sigma}{}_{,000}-\left[\overset{(p)}{d}{}^{\rho\rho}{}_{,000}\overset{(q)}{d}{}^{\phi\pi}{}_{,000}+\overset{(p)}{d}{}^{\phi\pi}{}_{,000}\overset{(q)}{d}{}^{\rho\rho}{}_{,000}\right.\\
&\left.\left.\left.\left.-4\overset{(p)}{d}{}^{\rho\phi}{}_{,000}\overset{(q)}{d}{}^{\rho\pi}{}_{,000}\right]n_\phi n_\pi-\overset{(p)}{d}{}^{\rho\sigma}{}_{,000}\overset{(q)}{d}{}^{\phi\pi}{}_{,000}n_\rho n_\sigma n_\phi n_\pi\right\}\right)\right\}\,d\Omega
\end{aligned}
\tag{3.17}
$$

where

$$d\Omega=r^{-2}\,dS \tag{3.18}$$

and

$$\overset{(i)}{d}{}^{\alpha\beta}\equiv\overset{(i)}{d}{}^{\alpha\beta}\left(t-r/c+\overset{(i)}{\underline{L}}.\underline{n}/c\right)\qquad i=p,q \tag{3.19}$$

Taking the ith radiator to be periodic (Booth, 1973) with frequency $\overset{(i)}{\omega}$, then

$$\overset{(i)}{d}{}^{\alpha\beta}=\operatorname{Re}\left\{\overset{(i)}{A}{}^{\alpha\beta}\exp i\left[\overset{(i)}{\omega}\left(t-r/c+\overset{(i)}{\underline{L}}.\underline{n}/c\right)+\overset{(i)}{\gamma}\right]\right\} \tag{3.20}$$

where $\overset{(i)}{A}{}^{\alpha\beta}$ is a complex amplitude and $\overset{(i)}{\gamma}$ is a phase angle. Thus

$$\overset{(i)}{d}{}^{\alpha\beta}=\operatorname{Re}\overset{(i)}{d}{}^{\alpha\beta}_{c} \tag{3.21}$$

and

$$\overset{(i)}{d}{}^{\alpha\beta}_{c}\left(t-r/c+\overset{(i)}{\underline{L}}.\underline{n}/c\right)=\left(\cos\overset{(i)}{\underline{k}}.\underline{n}+i\sin\overset{(i)}{\underline{k}}.\underline{n}\right)\overset{(i)}{d}{}^{\alpha\beta}_{c}(t-r/c) \tag{3.22}$$

where

$$\overset{(i)}{\underline{k}}\equiv\overset{(i)}{\omega}\overset{(i)}{\underline{L}}/c \tag{3.23}$$

When equations (3.20) to (3.23) are substituted into (3.17) for the pth and qth radiators, a typical product in the integrand of equation (3.17) then becomes.

$$
\begin{aligned}
\overset{(p)}{d}{}^{\alpha\beta}{}_{,000}\overset{(q)}{d}{}^{\gamma\delta}{}_{,000}&=\tfrac{1}{2}\left\{\overset{(p)}{d}{}^{\alpha\beta}{}_{,000}\overset{(q)}{d}{}^{\gamma\delta}{}_{,000}-\overset{(p)}{\bar{d}}{}^{\alpha\beta}{}_{,000}\overset{(q)}{\bar{d}}{}^{\gamma\delta}{}_{,000}\right\}\cos\left[\left(\overset{(p)}{\underline{k}}+\overset{(q)}{\underline{k}}\right).\underline{n}\right]\\
&+\tfrac{1}{2}\left\{\overset{(p)}{d}{}^{\alpha\beta}{}_{,000}\overset{(q)}{d}{}^{\gamma\delta}{}_{,000}+\overset{(p)}{\bar{d}}{}^{\alpha\beta}{}_{,000}\overset{(q)}{\bar{d}}{}^{\gamma\delta}{}_{,000}\right\}\cos\left[\left(\overset{(p)}{\underline{k}}-\overset{(q)}{\underline{k}}\right).\underline{n}\right]\\
&+\tfrac{1}{2}\left\{\overset{(p)}{\bar{d}}{}^{\alpha\beta}{}_{,000}\overset{(q)}{d}{}^{\gamma\delta}{}_{,000}+\overset{(p)}{d}{}^{\alpha\beta}{}_{,000}\overset{(q)}{\bar{d}}{}^{\gamma\delta}{}_{,000}\right\}\sin\left[\left(\overset{(p)}{\underline{k}}+\overset{(q)}{\underline{k}}\right).\underline{n}\right]\\
&+\tfrac{1}{2}\left\{\overset{(p)}{\bar{d}}{}^{\alpha\beta}{}_{,000}\overset{(q)}{d}{}^{\gamma\delta}{}_{,000}-\overset{(p)}{d}{}^{\alpha\beta}{}_{,000}\overset{(q)}{\bar{d}}{}^{\gamma\delta}{}_{,000}\right\}\sin\left[\left(\overset{(p)}{\underline{k}}-\overset{(q)}{\underline{k}}\right).\underline{n}\right]
\end{aligned}
\tag{3.24}
$$

where, on the right-hand side of equation (3.25)

$$\overset{(i)}{d}{}^{\alpha\beta} \equiv \overset{(i)}{d}{}^{\alpha\beta}(t - r/c)$$
$$\overset{(i)}{\bar{d}}{}^{\alpha\beta} \equiv \overset{(i)}{d}{}^{\alpha\beta}(t - r/c + \pi/2\overset{(i)}{\omega}) \qquad i = p, q \tag{3.25}$$

To facilitate the computation of equation (3.17)

$$\underline{\rho}_{pq} = \overset{(p)}{\underline{k}} + \overset{(q)}{\underline{k}} = \rho_{pq}\underline{\hat{n}}$$
$$^*\underline{\rho}_{pq} = \overset{(p)}{\underline{k}} - \overset{(q)}{\underline{k}} = {}^*\rho_{pq}{}^*\underline{\hat{n}} \tag{3.26}$$

and defined, where $\underline{\hat{n}}$ and $^*\underline{\hat{n}}$ are fixed unit vectors which specify the orientation of O_p and O_q relative to the common origin O.†

From equations (3.26) it is seen that

$$\underline{\rho}_{pq} \cdot \underline{n} = \rho_{pq}\hat{n}_\alpha n_\alpha$$
$$^*\underline{\rho}_{pq} \cdot \underline{n} = {}^*\rho_{pq}{}^*\hat{n}_\alpha n_\alpha \tag{3.27}$$

In order to evaluate the integral of equation (3.17) it is required to evaluate integrals of the form

$$\int_{4\pi} n_{\alpha_1} n_{\alpha_2} \ldots n_{\alpha_m} \exp(\rho_\omega n_\omega)\, d\Omega \tag{3.28}$$

and a detailed discussion of this integral is located in Appendix B. Equations (B.3.1) to (B.3.5), (3.20), (3.24), (3.25), (3.27) combined with equation (3.17) yield the time-averaged interaction angular momentum flux between the pth and qth radiators to quadrupole-quadrupole order as‡

$$\langle \dot{\mathscr{L}}^{\alpha\beta}{}_{pq} \rangle = -\frac{G}{c^4}\Bigg\{\Bigg(\overset{(p)}{d}{}^{\beta\phi}{}_{,00}\overset{(q)}{d}{}^{\alpha\phi}{}_{,000} - \overset{(p)}{d}{}^{\alpha\phi}{}_{,00}\overset{(q)}{d}{}^{\beta\phi}{}_{,000} - \overset{(p)}{\bar{d}}{}^{\beta\phi}{}_{,00}\overset{(q)}{\bar{d}}{}^{\alpha\phi}{}_{,000}$$
$$+ \overset{(p)}{\bar{d}}{}^{\alpha\phi}{}_{,00}\overset{(q)}{\bar{d}}{}^{\beta\phi}{}_{,000}\Bigg) \cdot (\sin \rho_{pq}[-\rho_{pq}^{-1} + 3\rho_{pq}^{-3} - 3\rho_{pq}^{-5}]$$
$$+ \cos \rho_{pq}[-2\rho_{pq}^{-2} + 3\rho_{pq}^{-4}]) + \Bigg(\overset{(p)}{d}{}^{\beta\phi}{}_{,00}\overset{(q)}{d}{}^{\alpha\phi}{}_{,000} - \overset{(p)}{d}{}^{\alpha\phi}{}_{,00}\overset{(q)}{d}{}^{\beta\phi}{}_{,000}$$
$$+ \overset{(p)}{\bar{d}}{}^{\beta\phi}{}_{,00}\overset{(q)}{\bar{d}}{}^{\alpha\phi}{}_{,000} - \overset{(p)}{\bar{d}}{}^{\alpha\phi}{}_{,00}\overset{(q)}{\bar{d}}{}^{\beta\phi}{}_{,000}\Bigg) \cdot (\sin {}^*\rho_{pq}[-{}^*\rho_{pq}^{-1} + 3{}^*\rho_{pq}^{-3}$$
$$- 3{}^*\rho_{pq}^{-5}] + \cos {}^*\rho_{pq}[-2{}^*\rho_{pq}^{-2} + 3{}^*\rho_{pq}^{-4}]) + \Bigg(\overset{(p)}{d}{}^{\beta\phi}{}_{,00}\overset{(q)}{d}{}^{\alpha\pi}{}_{,000}$$

† O_p and O_q are the centres of mass of the pth and qth radiators respectively.

‡ $\langle \dot{\mathscr{L}}^{\alpha\beta}_{pq} \rangle$ could be evaluated to higher multiple order by retention of higher order terms in equation (3.9).

$$-\overset{(p)}{d}{}^{\alpha\phi}{}_{,00}\overset{(q)}{d}{}^{\beta\pi}{}_{,000}-\overset{(p)}{\bar{d}}{}^{\beta\phi}{}_{,00}\overset{(q)}{\bar{d}}{}^{\alpha\pi}{}_{,000}+\overset{(p)}{\bar{d}}{}^{\alpha\phi}{}_{,00}\overset{(q)}{\bar{d}}{}^{\beta\pi}{}_{,000}\Big)\,\hat{n}_{\phi}\hat{n}_{\pi}$$

$$.\,(\sin\rho_{pq}\,[\rho_{pq}^{-1}-9\rho_{pq}^{-3}+15\rho_{pq}^{-5}]\;+\cos\rho_{pq}\,[4\rho_{pq}^{-2}-15\rho_{pq}^{-4}])$$

$$+\Big(\overset{(p)}{d}{}^{\beta\phi}{}_{,00}\overset{(q)}{\bar{d}}{}^{\alpha\pi}{}_{,000}-\overset{(p)}{d}{}^{\alpha\phi}{}_{,00}\overset{(q)}{d}{}^{\beta\pi}{}_{,000}+\overset{(p)}{\bar{d}}{}^{\beta\phi}{}_{,00}\overset{(q)}{\bar{d}}{}^{\alpha\pi}{}_{,000}$$

$$-\overset{(p)}{\bar{d}}{}^{\alpha\phi}{}_{,00}\overset{(q)}{\bar{d}}{}^{\beta\pi}{}_{,000}\Big){}^{*}\hat{n}_{\phi}{}^{*}\hat{n}_{\pi}\;.\,(\sin{}^{*}\rho_{pq}\,[{}^{*}\rho_{pq}^{-1}-9{}^{*}\rho_{pq}^{-3}+15{}^{*}\rho_{pq}^{-5}]$$

$$+\cos{}^{*}\rho_{pq}\,[4{}^{*}\rho_{pq}^{-2}-15{}^{*}\rho_{pq}^{-4}])+\Big[\Big(\overset{(p)}{d}{}^{\beta\phi}{}_{,00}\hat{n}_{\alpha}-\overset{(p)}{d}{}^{\alpha\phi}{}_{,00}\hat{n}_{\beta}\Big)\overset{(q)}{d}{}^{\sigma\sigma}{}_{,000}$$

$$+\Big(\overset{(q)}{d}{}^{\beta\phi}{}_{,00}\hat{n}_{\alpha}-\overset{(q)}{d}{}^{\alpha\phi}{}_{,00}\hat{n}_{\beta}\Big)\overset{(p)}{d}{}^{\sigma\sigma}{}_{,000}-\Big(\overset{(p)}{\bar{d}}{}^{\beta\phi}{}_{,00}\hat{n}_{\alpha}-\overset{(p)}{\bar{d}}{}^{\alpha\phi}{}_{,00}\hat{n}_{\beta}\Big)\,\overset{(q)}{\bar{d}}{}^{\sigma\sigma}{}_{,000}$$

$$-\Big(\overset{(q)}{\bar{d}}{}^{\beta\phi}{}_{,00}\hat{n}_{\alpha}-\overset{(q)}{\bar{d}}{}^{\alpha\phi}{}_{,00}\hat{n}_{\beta}\Big)\overset{(p)}{d}{}^{\sigma\sigma}{}_{,000}\Big]\hat{n}_{\phi}\,.\,(\sin\rho_{pq}\,[-\tfrac{1}{4}\rho_{pq}^{-1}-\tfrac{3}{4}\rho_{pq}^{-3}$$

$$+\tfrac{15}{4}\rho_{pq}^{-5}]+\cos\rho_{pq}\,[-\tfrac{1}{2}\rho_{pq}^{-2}-\tfrac{15}{4}\rho_{pq}^{-4}])-\Big[\Big(\overset{(p)}{d}{}^{\beta\phi}{}_{,00}\hat{n}_{\alpha}$$

$$-\overset{(p)}{d}{}^{\alpha\phi}{}_{,00}\hat{n}_{\beta}\Big)\overset{(q)}{d}{}^{\sigma\sigma}{}_{,000}-\Big(\overset{(q)}{d}{}^{\beta\phi}{}_{,00}\hat{n}_{\alpha}-\overset{(q)}{d}{}^{\alpha\phi}{}_{,00}\hat{n}_{\beta}\Big)\overset{(p)}{d}{}^{\sigma\sigma}{}_{,000}$$

$$+\Big(\overset{(p)}{\bar{d}}{}^{\beta\phi}{}_{,00}\hat{n}_{\alpha}-\overset{(p)}{\bar{d}}{}^{\alpha\phi}{}_{,00}\hat{n}_{\beta}\Big)\,\overset{(q)}{\bar{d}}{}^{\sigma\sigma}{}_{,000}-\Big(\overset{(q)}{\bar{d}}{}^{\beta\phi}{}_{,00}\hat{n}_{\alpha}-\overset{(q)}{\bar{d}}{}^{a\phi}{}_{,00}\hat{n}_{\beta}\Big)$$

$$\times\overset{(p)}{\bar{d}}{}^{\sigma\sigma}{}_{,000}\Big]{}^{*}\hat{n}_{\phi}\;.\,(\sin{}^{*}\rho_{pq}\,[-\tfrac{1}{4}{}^{*}\rho_{pq}^{-2}-\tfrac{3}{4}{}^{*}\rho_{pq}^{-4}+\tfrac{15}{4}{}^{*}\rho_{pq}^{-6}]$$

$$+\cos{}^{*}\rho_{pq}\,[-\tfrac{1}{2}{}^{*}\rho_{pq}^{-3}-\tfrac{15}{4}{}^{*}\rho_{pq}^{-5}])\rho_{pq}+\Big[\Big(\overset{(p)}{d}{}^{\beta\phi}{}_{,00}{}^{*}\hat{n}_{\alpha}-\overset{(p)}{d}{}^{\alpha\phi}{}_{,00}{}^{*}\hat{n}_{\beta}\Big)$$

$$\times\overset{(q)}{d}{}^{\sigma\sigma}{}_{,000}+\Big(\overset{(q)}{d}{}^{\beta\phi}{}_{,00}{}^{*}\hat{n}_{\alpha}-\overset{(q)}{d}{}^{\alpha\phi}{}_{,00}{}^{*}\hat{n}_{\beta}\Big)\overset{(p)}{d}{}^{\sigma\sigma}{}_{,000}$$

$$+\Big(\overset{(p)}{\bar{d}}{}^{\beta\phi}{}_{,00}{}^{*}\hat{n}_{\alpha}-\overset{(p)}{\bar{d}}{}^{\alpha\phi}{}_{,00}{}^{*}\hat{n}_{\beta}\Big)\overset{(q)}{\bar{d}}{}^{\sigma\sigma}{}_{,000}-\Big(\overset{(q)}{\bar{d}}{}^{\beta\phi}{}_{,00}{}^{*}\hat{n}_{\alpha}$$

$$-\overset{(q)}{\bar{d}}{}^{\alpha\phi}{}_{,00}{}^{*}\hat{n}_{\beta}\Big)\overset{(p)}{\bar{d}}{}^{\sigma\sigma}{}_{,000}\Big]{}^{*}\hat{n}_{\phi}\;.\,(\sin{}^{*}\rho_{pq}\,[-\tfrac{1}{4}{}^{*}\rho_{pq}^{-1}-\tfrac{3}{4}{}^{*}\rho_{pq}^{-3}+\tfrac{15}{4}{}^{*}\rho_{pq}^{-5}]$$

$$+\cos{}^{*}\rho_{pq}\,[-\tfrac{1}{2}{}^{*}\rho_{pq}^{-2}-\tfrac{15}{4}{}^{*}\rho_{pq}^{-4}])-\Big[\Big(\overset{(p)}{d}{}^{\beta\phi}{}_{,00}{}^{*}\hat{n}_{\alpha}-\overset{(p)}{d}{}^{\alpha\phi}{}_{,00}{}^{*}\hat{n}_{\beta}\Big)$$

$$\times \overset{(q)}{d}{}^{\sigma\sigma}{}_{,000} - \Big(\overset{(q)}{d}{}^{\beta\phi}{}_{,00}{}^*\hat{n}_\alpha - \overset{(q)}{d}{}^{\alpha\phi}{}_{,00}{}^*\hat{n}_\beta\Big) \overset{(p)}{d}{}^{\sigma\sigma}{}_{,000}$$

$$- \Big(\overset{(p)}{\bar{d}}{}^{\beta\phi}{}_{,00}{}^*\hat{n}_\alpha - \overset{(p)}{\bar{d}}{}^{\alpha\phi}{}_{,00}{}^*\hat{n}_\beta\Big) \overset{(q)}{\bar{d}}{}^{\sigma\sigma}{}_{,000} + \Big(\overset{(q)}{\bar{d}}{}^{\beta\phi}{}_{,00}{}^*\hat{n}_\alpha$$

$$- \overset{(q)}{\bar{d}}{}^{\alpha\phi}{}_{,00}\hat{n}_\beta\Big) \overset{(p)}{\bar{d}}{}^{\sigma\sigma}{}_{,000}\Big] \hat{n}_\phi \,.\, (\sin \rho_{pq}\,[-\tfrac{1}{4}\rho_{pq}^{-2} - \tfrac{3}{4}\rho_{pq}^{-4} + \tfrac{15}{4}\rho_{pq}^{-6}]$$

$$+ \cos \rho_{pq}\,[-\tfrac{1}{2}\rho_{pq}^{-3} - \tfrac{15}{4}\rho_{pq}^{-5}])^*\rho_{pq} + \Big[\Big(\overset{(p)}{d}{}^{\beta\phi}{}_{,00}\hat{n}_\alpha - \overset{(p)}{d}{}^{\alpha\phi}{}_{,00}\hat{n}_\beta\Big) \overset{(q)}{d}{}^{\phi\pi}{}_{,000}$$

$$+ \Big(\overset{(q)}{d}{}^{\beta\phi}{}_{,00}\hat{n}_\alpha - \overset{(q)}{d}{}^{\alpha\phi}{}_{,00}\hat{n}_\beta\Big) \overset{(p)}{d}{}^{\phi\pi}{}_{,000} - \Big(\overset{(p)}{\bar{d}}{}^{\beta\phi}{}_{,00}\hat{n}_\alpha - \overset{(p)}{\bar{d}}{}^{\alpha\phi}{}_{,00}\hat{n}_\beta\Big) \overset{(q)}{\bar{d}}{}^{\phi\pi}{}_{,000}$$

$$- \Big(\overset{(q)}{\bar{d}}{}^{\beta\phi}{}_{,00}\hat{n}_\alpha - \overset{(q)}{\bar{d}}{}^{\alpha\phi}{}_{,00}\hat{n}_\beta\Big) \overset{(p)}{\bar{d}}{}^{\phi\pi}{}_{,000}\Big] \hat{n}_\pi \,.\, (\sin \rho_{pq}\,[\tfrac{1}{2}\rho_{pq}^{-1} - \tfrac{9}{2}\rho_{pq}^{-3}$$

$$+ \tfrac{15}{2}\rho_{pq}^{-5}] + \cos \rho_{pq}\,[2\rho_{pq}^{-2} - \tfrac{15}{2}\rho_{pq}^{-4}]) - \Big[\Big(\overset{(p)}{d}{}^{\beta\phi}{}_{,00}\hat{n}_\alpha - \overset{(p)}{d}{}^{\alpha\phi}{}_{,00}\hat{n}_\beta\Big) \overset{(q)}{d}{}^{\phi\pi}{}_{,000}$$

$$- \Big(\overset{(q)}{d}{}^{\beta\phi}{}_{,00}\hat{n}_\alpha - \overset{(q)}{d}{}^{\alpha\phi}{}_{,00}\hat{n}_\beta\Big) \overset{(p)}{d}{}^{\phi\pi}{}_{,000} + \Big(\overset{(p)}{\bar{d}}{}^{\beta\phi}{}_{,00}\hat{n}_\alpha - \overset{(p)}{\bar{d}}{}^{\alpha\phi}{}_{,00}\hat{n}_\beta\Big) \overset{(q)}{\bar{d}}{}^{\phi\pi}{}_{,000}$$

$$- \Big(\overset{(q)}{\bar{d}}{}^{\beta\phi}{}_{,00}\hat{n}_\alpha - \overset{(q)}{\bar{d}}{}^{\alpha\phi}{}_{,00}\hat{n}_\beta\Big) \overset{(p)}{\bar{d}}{}^{\phi\pi}{}_{,000}\Big] {}^*\hat{n}_\pi \,.\, (\sin {}^*\rho_{pq}\,[\tfrac{1}{2}{}^*\rho_{pq}^{-2} - \tfrac{9}{2}{}^*\rho_{pq}^{-4}$$

$$+ \tfrac{15}{2}{}^*\rho_{pq}^{-6}] + \cos {}^*\rho_{pq}\,[2{}^*\rho_{pq}^{-3} - \tfrac{15}{2}{}^*\rho_{pq}^{-5}])\rho_{pq} + \Big[\Big(\overset{(p)}{d}{}^{\beta\phi}{}_{,00}{}^*\hat{n}_\alpha$$

$$- \overset{(p)}{d}{}^{\alpha\phi}{}_{,00}{}^*\hat{n}_\beta\Big) \overset{(q)}{d}{}^{\phi\pi}{}_{,000} + \Big(\overset{(q)}{d}{}^{\beta\phi}{}_{,00}{}^*\hat{n}_\alpha - \overset{(q)}{d}{}^{\alpha\phi}{}_{,00}{}^*\hat{n}_\beta\Big) \overset{(p)}{d}{}^{\phi\pi}{}_{,000}$$

$$+ \Big(\overset{(p)}{\bar{d}}{}^{\beta\phi}{}_{,00}{}^*\hat{n}_\alpha - \overset{(p)}{\bar{d}}{}^{\alpha\phi}{}_{,00}{}^*\hat{n}_\beta\Big) \overset{(q)}{\bar{d}}{}^{\phi\pi}{}_{,000} + \Big(\overset{(q)}{\bar{d}}{}^{\beta\phi}{}_{,00}{}^*\hat{n}_\alpha$$

$$- \overset{(q)}{\bar{d}}{}^{\alpha\phi}{}_{,00}{}^*\hat{n}_\beta\Big) \overset{(p)}{\bar{d}}{}^{\phi\pi}{}_{,000}\Big] {}^*\hat{n}_\pi \,.\, (\sin {}^*\rho_{pq}\,[\tfrac{1}{2}{}^*\rho_{pq}^{-1} - \tfrac{9}{2}{}^*\rho_{pq}^{-3} + \tfrac{15}{2}{}^*\rho_{pq}^{-5}]$$

$$+ \cos {}^*\rho_{pq}\,[2{}^*\rho_{pq}^{-2} - \tfrac{15}{2}{}^*\rho_{pq}^{-4}]) - \Big[\Big(\overset{(p)}{d}{}^{\beta\phi}{}_{,00}{}^*\hat{n}_\alpha - \overset{(p)}{d}{}^{\alpha\phi}{}_{,00}{}^*\hat{n}_\beta\Big) \overset{(q)}{d}{}^{\phi\pi}{}_{,000}$$

$$- \Big(\overset{(q)}{d}{}^{\beta\phi}{}_{,00}{}^*\hat{n}_\alpha - \overset{(q)}{d}{}^{\alpha\phi}{}_{,00}{}^*\hat{n}_\beta\Big) \overset{(p)}{d}{}^{\phi\pi}{}_{,000} - \Big(\overset{(p)}{\bar{d}}{}^{\beta\phi}{}_{,00}{}^*\hat{n}_\alpha - \overset{(p)}{\bar{d}}{}^{\alpha\phi}{}_{,00}{}^*\hat{n}_\beta\Big)$$

$$\times \overset{(q)}{d}{}^{\phi\pi}{}_{,000} + \Big(\overset{(q)}{d}{}^{\beta\phi}{}_{,00}{}^*\hat{n}_\alpha - \overset{(q)}{\bar{d}}{}^{\alpha\phi}{}_{,00}{}^*\hat{n}_\beta\Big) \overset{(p)}{d}{}^{\phi\pi}{}_{,000}\Big] \hat{n}_\pi$$

$$. (\sin \rho_{pq} [\tfrac{1}{2}\rho_{pq}^{-2} - \tfrac{9}{2}\rho_{pq}^{-4} + \tfrac{15}{2}\rho_{pq}^{-6}] + \cos \rho_{pq} [2\rho_{pq}^{-3} - \tfrac{15}{2}\rho_{pq}^{-5}])^{*}\rho_{pq}$$

$$+ \left[\left(\overset{(p)}{d}{}^{\beta\phi}{}_{,00}\hat{n}_{\alpha} - \overset{(p)}{d}{}^{\alpha\phi}{}_{,00}\hat{n}_{\beta}\right)\overset{(q)}{d}{}^{\rho\sigma}{}_{,000} + \left(\overset{(q)}{d}{}^{\beta\phi}{}_{,00}\hat{n}_{\alpha} - \overset{(q)}{d}{}^{\alpha\phi}{}_{,00}\hat{n}_{\beta}\right)\overset{(p)}{d}{}^{\rho\sigma}{}_{,000}\right.$$

$$\left. - \left(\overset{(p)}{\bar{d}}{}^{\beta\phi}{}_{,00}\hat{n}_{\alpha} - \overset{(p)}{\bar{d}}{}^{\alpha\phi}{}_{,00}\hat{n}_{\beta}\right)\overset{(q)}{\bar{d}}{}^{\rho\sigma}{}_{,000} - \left(\overset{(q)}{\bar{d}}{}^{\beta\phi}{}_{,00}\hat{n}_{\alpha} - \overset{(q)}{\bar{d}}{}^{\alpha\phi}{}_{,00}\hat{n}_{\beta}\right)\overset{(p)}{\bar{d}}{}^{\rho\sigma}{}_{,000}\right]$$

$$\times \hat{n}_{\phi}\hat{n}_{\rho}\hat{n}_{\sigma} \; .(\sin \rho_{pq} [-\tfrac{1}{4}\rho_{pq}^{-1} + \tfrac{45}{4}\rho_{pq}^{-3} - \tfrac{105}{4}\rho_{pq}^{-5}]$$

$$+ \cos \rho_{pq} [-\tfrac{5}{2}\rho_{pq}^{-2} + \tfrac{105}{4}\rho_{pq}^{-4}]) - \left[\left(\overset{(p)}{d}{}^{\beta\phi}{}_{,00}\hat{n}_{\alpha} - \overset{(p)}{d}{}^{\alpha\phi}{}_{,00}\hat{n}_{\beta}\right)\overset{(q)}{d}{}^{\rho\sigma}{}_{,000}\right.$$

$$- \left(\overset{(q)}{d}{}^{\beta\phi}{}_{,00}\hat{n}_{\alpha} - \overset{(q)}{d}{}^{\alpha\phi}{}_{,00}\hat{n}_{\beta}\right)\overset{(q)}{d}{}^{\rho\sigma}{}_{,000} + \left(\overset{(p)}{\bar{d}}{}^{\beta\phi}{}_{,00}\hat{n}_{\alpha} - \overset{(p)}{\bar{d}}{}^{\alpha\phi}{}_{,00}\hat{n}_{\beta}\right)\overset{(q)}{\bar{d}}{}^{\rho\sigma}{}_{,000}$$

$$\left. - \left(\overset{(q)}{\bar{d}}{}^{\beta\phi}{}_{,00}\hat{n}_{\alpha} - \overset{(q)}{\bar{d}}{}^{\alpha\phi}{}_{,00}\hat{n}_{\beta}\right)\overset{(p)}{\bar{d}}{}^{\rho\sigma}{}_{,000}\right] {}^{*}\hat{n}_{\phi}{}^{*}\hat{n}_{\rho}{}^{*}\hat{n}_{\sigma}$$

$$.(\sin {}^{*}\rho_{pq} [-\tfrac{1}{4}{}^{*}\rho_{pq}^{-2} + \tfrac{45}{4}{}^{*}\rho_{pq}^{-4} - \tfrac{105}{4}{}^{*}\rho_{pq}^{-6}] + \cos {}^{*}\rho_{pq} [-\tfrac{5}{2}{}^{*}\rho_{pq}^{-3}$$

$$+ \tfrac{105}{4}{}^{*}\rho_{pq}^{-5}])\rho_{pq} + \left[\left(\overset{(p)}{d}{}^{\beta\phi}{}_{,00}{}^{*}\hat{n}_{\alpha} - \overset{(p)}{d}{}^{\alpha\phi}{}_{,00}{}^{*}\hat{n}_{\beta}\right)\overset{(q)}{d}{}^{\rho\sigma}{}_{,000}\right.$$

$$+ \left(\overset{(q)}{d}{}^{\beta\phi}{}_{,00}{}^{*}\hat{n}_{\alpha} - \overset{(q)}{d}{}^{\alpha\phi}{}_{,00}{}^{*}\hat{n}_{\beta}\right)\overset{(p)}{d}{}^{\rho\sigma}{}_{,000} + \left(\overset{(p)}{\bar{d}}{}^{\beta\phi}{}_{,00}{}^{*}\hat{n}_{\alpha}\right.$$

$$\left.\left. - \overset{(p)}{\bar{d}}{}^{\alpha\phi}{}_{,00}{}^{*}\hat{n}_{\beta}\right)\overset{(q)}{\bar{d}}{}^{\rho\sigma}{}_{,000} + \left(\overset{(q)}{\bar{d}}{}^{\beta\phi}{}_{,00}{}^{*}\hat{n}_{\alpha} - \overset{(q)}{\bar{d}}{}^{\alpha\phi}{}_{,00}{}^{*}\hat{n}_{\beta}\right)\overset{(p)}{\bar{d}}{}^{\rho\sigma}{}_{,000}\right]$$

$$\times {}^{*}\hat{n}_{\phi}{}^{*}\hat{n}_{\rho}{}^{*}\hat{n}_{\sigma} \; .(\sin {}^{*}\rho_{pq} [-\tfrac{1}{4}{}^{*}\rho_{pq}^{-1} + \tfrac{45}{4}{}^{*}\rho_{pq}^{-3} - \tfrac{105}{4}{}^{*}\rho_{pq}^{-5}]$$

$$+ \cos {}^{*}\rho_{pq} [-\tfrac{5}{2}{}^{*}\rho_{pq}^{-2} + \tfrac{105}{4}{}^{*}\rho_{pq}^{-4}]) - \left[\left(\overset{(p)}{d}{}^{\beta\phi}{}_{,00}{}^{*}\hat{n}_{\alpha} - \overset{(p)}{d}{}^{\alpha\phi}{}_{,00}{}^{*}\hat{n}_{\beta}\right)\overset{(q)}{d}{}^{\rho\sigma}{}_{,000}\right.$$

$$- \left(\overset{(q)}{d}{}^{\beta\phi}{}_{,00}{}^{*}\hat{n}_{\alpha} - \overset{(q)}{d}{}^{\alpha\phi}{}_{,00}{}^{*}\hat{n}_{\beta}\right)\overset{(p)}{d}{}^{\rho\sigma}{}_{,000} - \left(\overset{(p)}{\bar{d}}{}^{\beta\phi}{}_{,00}{}^{*}\hat{n}_{\alpha} - \overset{(p)}{\bar{d}}{}^{\alpha\phi}{}_{,00}{}^{*}\hat{n}_{\beta}\right)$$

$$\left.\times \overset{(q)}{\bar{d}}{}^{\rho\sigma}{}_{,000} + \left(\overset{(q)}{\bar{d}}{}^{\beta\phi}{}_{,00}{}^{*}\hat{n}_{\alpha} - \overset{(q)}{\bar{d}}{}^{\alpha\phi}{}_{,00}{}^{*}\hat{n}_{\beta}\right)\overset{(p)}{\bar{d}}{}^{\rho\sigma}{}_{,000}\right] \hat{n}_{\phi}\hat{n}_{\rho}\hat{n}_{\sigma}$$

$$.(\sin \rho_{pq} [-\tfrac{1}{4}\rho_{pq}^{-2} + \tfrac{45}{4}\rho_{pq}^{-4} - \tfrac{105}{4}\rho_{pq}^{-6}] + \cos \rho_{pq} [-\tfrac{5}{2}\rho_{pq}^{-3} + \tfrac{105}{4}\rho_{pq}^{-5}])$$

$$\times\ {}^*\rho_{pq} + \left(\overset{(p)}{d}{}^{\phi\phi}{}_{,00}\overset{(q)}{d}{}^{\pi\pi}{}_{,000} - \overset{(p)}{\bar{d}}{}^{\phi\phi}{}_{,00}\overset{(q)}{\bar{d}}{}^{\pi\pi}{}_{,000}\right)(\hat{n}_\alpha{}^*\hat{n}_\beta - {}^*\hat{n}_\alpha\hat{n}_\beta){}^*\rho_{pq}$$

$$.\,(\sin\rho_{pq}\,[-\tfrac{3}{8}\rho_{pq}^{-2} + \tfrac{15}{8}\rho_{pq}^{-4}] + \cos\rho_{pq}\,[\tfrac{1}{8}\rho_{pq}^{-1} - \tfrac{5}{8}\rho_{pq}^{-3} - \tfrac{15}{8}\rho_{pq}^{-5}])$$

$$+\left(\overset{(p)}{d}{}^{\phi\phi}{}_{,00}\overset{(q)}{d}{}^{\pi\pi}{}_{,000} + \overset{(p)}{\bar{d}}{}^{\phi\phi}{}_{,00}\overset{(q)}{\bar{d}}{}^{\pi\pi}{}_{,000}\right)({}^*\hat{n}_\alpha\hat{n}_\beta - \hat{n}_\alpha{}^*\hat{n}_\beta)\rho_{pq}$$

$$.\,(\sin{}^*\rho_{pq}\,[-\tfrac{3}{8}{}^*\rho_{pq}^{-2} + \tfrac{15}{8}{}^*\rho_{pq}^{-4}] + \cos{}^*\rho_{pq}\,[\tfrac{1}{8}{}^*\rho_{pq}^{-1} - \tfrac{5}{8}{}^*\rho_{pq}^{-3} - \tfrac{15}{8}{}^*\rho_{pq}^{-5}])$$

$$+\left(\overset{(p)}{d}{}^{\phi\pi}{}_{,00}\overset{(q)}{d}{}^{\phi\pi}{}_{,000} - \overset{(p)}{\bar{d}}{}^{\phi\pi}{}_{,00}\overset{(q)}{\bar{d}}{}^{\phi\pi}{}_{,000}\right)(\hat{n}_\alpha{}^*\hat{n}_\beta - {}^*\hat{n}_\alpha\hat{n}_\beta)\,{}^*\rho_{pq}$$

$$.\,(\sin\rho_{pq}\,[\tfrac{3}{4}\rho_{pq}^{-2} - 3\rho_{pq}^{-4} + \tfrac{15}{4}\rho_{pq}^{-6}] + \cos\rho_{pq}\,[-\tfrac{1}{4}\rho_{pq}^{-1} + \tfrac{7}{4}\rho_{pq}^{-3} - \tfrac{15}{4}\rho_{pq}^{-5}])$$

$$+\left(\overset{(p)}{d}{}^{\phi\pi}{}_{,00}\overset{(q)}{d}{}^{\phi\pi}{}_{,000} + \overset{(p)}{\bar{d}}{}^{\phi\pi}{}_{,00}\overset{(q)}{\bar{d}}{}^{\phi\pi}{}_{,000}\right)({}^*\hat{n}_\alpha\hat{n}_\beta - \hat{n}_\alpha{}^*\hat{n}_\beta)\rho_{pq}$$

$$.\,(\sin{}^*\rho_{pq}\,[\tfrac{3}{4}{}^*\rho_{pq}^{-2} - 3{}^*\rho_{pq}^{-4} + \tfrac{15}{4}{}^*\rho_{pq}^{-6}] + \cos{}^*\rho_{pq}\,[-\tfrac{1}{4}{}^*\rho_{pq}^{-1}$$

$$+\tfrac{7}{4}{}^*\rho_{pq}^{-3} - \tfrac{15}{4}{}^*\rho_{pq}^{-5}]) + \left(\overset{(p)}{d}{}^{\sigma\phi}{}_{,00}\overset{(q)}{d}{}^{\sigma\pi}{}_{,000} - \overset{(p)}{\bar{d}}{}^{\sigma\phi}{}_{,00}\overset{(q)}{\bar{d}}{}^{\sigma\pi}{}_{,000}\right)$$

$$\times(\hat{n}_\alpha{}^*\hat{n}_\beta - {}^*\hat{n}_\alpha\hat{n}_\beta)\hat{n}_\phi\hat{n}_\pi{}^*\rho_{pq}\,.\,(\sin\rho_{pq}\,[-\tfrac{7}{2}\rho_{pq}^{-2} + 30\rho_{pq}^{-4} - \tfrac{105}{2}\rho_{pq}^{-6}]$$

$$+\cos\rho_{pq}\,[\tfrac{1}{2}\rho_{pq}^{-1} - \tfrac{25}{2}\rho_{pq}^{-3} + \tfrac{105}{2}\rho_{pq}^{-5}]) + \left(\overset{(p)}{d}{}^{\sigma\phi}{}_{,00}\overset{(q)}{d}{}^{\sigma\pi}{}_{,000}\right.$$

$$\left.+\overset{(p)}{\bar{d}}{}^{\sigma\phi}{}_{,00}\overset{(q)}{\bar{d}}{}^{\sigma\pi}{}_{,000}\right)({}^*\hat{n}_\alpha\hat{n}_\beta - \hat{n}_\alpha{}^*\hat{n}_\beta){}^*\hat{n}_\phi{}^*\hat{n}_\pi\rho_{pq}$$

$$.\,(\sin{}^*\rho_{pq}\,[-\tfrac{7}{2}{}^*\rho_{pq}^{-2} + 30{}^*\rho_{pq}^{-4} - \tfrac{105}{2}{}^*\rho_{pq}^{-6}] + \cos{}^*\rho_{pq}\,[\tfrac{1}{2}{}^*\rho_{pq}^{-1}$$

$$-\tfrac{25}{2}{}^*\rho_{pq}^{-3} + \tfrac{105}{2}{}^*\rho_{pq}^{-5}]) + \left(\overset{(p)}{d}{}^{\rho\sigma}{}_{,00}\overset{(q)}{d}{}^{\phi\pi}{}_{,000} - \overset{(p)}{\bar{d}}{}^{\rho\sigma}{}_{,00}\overset{(q)}{d}{}^{\phi\pi}{}_{,000}\right)$$

$$\times(\hat{n}_\alpha{}^*\hat{n}_\beta - {}^*\hat{n}_\alpha\hat{n}_\beta)\hat{n}_\rho\hat{n}_\sigma\hat{n}_\phi\hat{n}_\pi{}^*\rho_{pq}\,.\,(\sin\rho_{pq}\,[\tfrac{15}{8}\rho_{pq}^{-2} - \tfrac{105}{2}\rho_{pq}^{-4}$$

$$+\tfrac{945}{8}\rho_{pq}^{-6}] + \cos\rho_{pq}\,[-\tfrac{1}{8}\rho_{pq}^{-1} + \tfrac{105}{8}\rho_{pq}^{-3} - \tfrac{945}{8}\rho_{pq}^{-5}])$$

$$+\left(\overset{(p)}{d}{}^{\rho\sigma}{}_{,00}\overset{(q)}{d}{}^{\phi\pi}{}_{,000} + \overset{(p)}{\bar{d}}{}^{\rho\sigma}{}_{,00}\overset{(q)}{\bar{d}}{}^{\phi\pi}{}_{,000}\right)({}^*\hat{n}_\alpha\hat{n}_\beta - \hat{n}_\alpha{}^*\hat{n}_\beta)$$

$$\times\ {}^*\hat{n}_\rho {}^*\hat{n}_\sigma {}^*\hat{n}_\phi {}^*\hat{n}_\pi \rho_{pq} . (\sin {}^*\rho_{pq} [\tfrac{15}{8} {}^*\rho_{pq}^{-2} - \tfrac{105}{2} {}^*\rho_{pq}^{-4} + \tfrac{945}{8} {}^*\rho_{pq}^{-6}]$$

$$+ \cos {}^*\rho_{pq} [-\tfrac{1}{8} {}^*\rho_{pq}^{-1} + \tfrac{105}{8} {}^*\rho_{pq}^{-3} - \tfrac{945}{8} {}^*\rho_{pq}^{-5}]) + \Big(\overset{(p)}{d}{}^{\rho\rho}{}_{,00} \overset{(q)}{d}{}^{\phi\pi}{}_{,000}$$

$$- \overset{(p)}{\bar{d}}{}^{\rho\rho}{}_{,00} \overset{(q)}{d}{}^{\phi\pi}{}_{,000} - \overset{(p)}{\bar{d}}{}^{\rho\rho}{}_{,00} \overset{(q)}{\bar{d}}{}^{\phi\pi}{}_{,000} + \overset{(q)}{\bar{d}}{}^{\rho\rho}{}_{,00} \overset{(p)}{\bar{d}}{}^{\phi\pi}{}_{,000} \Big)$$

$$\times (\hat{n}_\alpha {}^*\hat{n}_\beta - {}^*\hat{n}_\alpha \hat{n}_\beta) \hat{n}_\phi \hat{n}_\pi . {}^*\rho_{pq} (\sin \rho_{pq} [\tfrac{5}{8} \rho_{pq}^{-2} + \tfrac{15}{4} \rho_{pq}^{-4} - \tfrac{105}{8} \rho_{pq}^{-6}]$$

$$+ \cos \rho_{pq} [-\tfrac{1}{8} \rho_{pq}^{-1} + \tfrac{5}{8} \rho_{pq}^{-3} + \tfrac{105}{8} \rho_{pq}^{-5}]) + \Big(\overset{(p)}{\bar{d}}{}^{\rho\rho}{}_{,00} \overset{(q)}{\bar{d}}{}^{\phi\pi}{}_{,000} - \overset{(p)}{\bar{d}}{}^{\rho\rho}{}_{,00}$$

$$\times \overset{(q)}{d}{}^{\phi\pi}{}_{,000} + \overset{(p)}{\bar{d}}{}^{\rho\rho}{}_{,00} \overset{(q)}{\bar{d}}{}^{\phi\pi}{}_{,000} - \overset{(q)}{\bar{d}}{}^{\rho\rho}{}_{,00} \overset{(p)}{\bar{d}}{}^{\phi\pi}{}_{,000} \Big) ({}^*\hat{n}_\alpha \hat{n}_\beta - \hat{n}_\alpha {}^*\hat{n}_\beta)$$

$$\times\ {}^*\hat{n}_\phi {}^*\hat{n}_\pi . \rho_{pq} (\sin {}^*\rho_{pq} [\tfrac{5}{8} {}^*\rho_{pq}^{-2} + \tfrac{15}{4} {}^*\rho_{pq}^{-4} - \tfrac{105}{8} {}^*\rho_{pq}^{-6}] + \cos {}^*\rho_{pq}$$

$$\times\ [-\tfrac{1}{8} {}^*\rho_{pq}^{-1} + \tfrac{5}{8} {}^*\rho_{pq}^{-3} + \tfrac{105}{8} {}^*\rho_{pq}^{-5}]) \Big\} \tag{3.29}$$

The total interaction angular momentum flux from a galaxy of N such radiators is then

$$\langle {}_{\text{tot}} \dot{\mathscr{L}}^{\alpha\beta} \rangle = \tfrac{1}{2} \sum_{\substack{p,q=1 \\ p \neq q}}^{N} \langle \dot{\mathscr{L}}^{\alpha\beta}{}_{pq} \rangle \tag{3.30}$$

If, in equation (3.29) the limit is taken as $\rho_{pq} \to 0$ and ${}^*\rho_{pq} \to 0$, then

$$\lim_{\substack{\rho_{pq} \\ {}^*\rho_{pq}} \Big\} \to 0} \langle \dot{\mathscr{L}}^{\alpha\beta}{}_{pq} \rangle = \frac{4G}{5c^4} \Big\{ \overset{(p)}{d}{}^{\beta\gamma}{}_{,00} \overset{(q)}{d}{}^{\alpha\gamma}{}_{,000} - \overset{(p)}{d}{}^{\alpha\gamma}{}_{,00} \overset{(q)}{d}{}^{\beta\gamma}{}_{,000} \Big\} \tag{3.31}$$

letting $p = q$ in equation (3.31) yields twice the angular momentum flux from the pth radiator in the absence of interaction (Cooperstock & Booth, 1969a). Consequently, the total angular momentum loss rate for this galactic model is

$$\langle {}_{\text{tot}} \dot{\mathscr{L}}^{\alpha\beta} \rangle = \tfrac{1}{2} \sum_{p,q=1}^{N} \langle \dot{\mathscr{L}}^{\alpha\beta}{}_{pq} \rangle \tag{3.32}$$

where

$$\langle \dot{\mathscr{L}}^{\alpha\beta}{}_{pp} \rangle = \lim_{\substack{\rho_{pq} \\ {}^*\rho_{pq}} \Big\} \to 0} \langle \dot{\mathscr{L}}^{\alpha\beta}{}_{pq} \rangle \Big|_{p=q} \tag{3.33}$$

Equation (3.29) is fully consistent with earlier work by Booth *et al.* (1972) in which stress-energy distributions were restricted to have their centres of mass lying on the common z-axes.

4. *Interaction Linear Momentum Flux*

If i is set equal to $\lambda(\lambda = 1, 2, 3)$ in equation (2.6) and the integration is performed over the hypersurface x^0 = constant, an application of the Gauss theorem yields the total linear momentum flux (Cooperstock & Booth, 1969b)

$$\dot{P}_\lambda = -\oint_S (-g) t_\lambda{}^\alpha n_\alpha \, dS \tag{4.1}$$

where S is the surface bounding the radiator-pair volume. This is in complete analogy with the energy loss-rate of Booth (1973). As in Section 3, equations (3.1), (3.2) and (3.4) allow the division

$$\dot{P}_\lambda = {}_p\dot{P}_\lambda + {}_q\dot{P}_\lambda + {}_{\text{int}}\dot{P}_\lambda \tag{4.2}$$

where ${}_p\dot{P}_\lambda$ and ${}_q\dot{P}_\lambda$ are the linear momentum fluxes from the radiator-pair in the absence of interaction and ${}_{\text{int}}\dot{P}_\lambda$ is the flux which arises from their interaction. It is of interest to note that

$${}_p\dot{P}_\lambda = {}_q\dot{P}_\lambda = 0 \tag{4.3}$$

to quadrupole-quadrupole order (Bonnor & Rotenburg, 1961, 1965; Papapetrou, 1962; Peres, 1962). However, to this order ${}_{\text{int}}\dot{P}_\lambda \neq 0$ and it is this term which is now derived.

Proceeding as in Section 3, equations (3.1), (3.2), (3.16) and (3.24) combined with (4.1) and (4.2) yield

$${}_{\text{int}}\dot{P}_\lambda = \frac{G}{76c^6} \int_{4\pi} \left\{ \overset{(p)}{D}{}^{\alpha\beta}{}_{,000} \overset{(q)}{D}{}^{\gamma\delta}{}_{,000} n_\alpha n_\beta n_\gamma n_\delta - 4\overset{(p)}{D}{}^{\alpha\gamma}{}_{,000} \overset{(q)}{D}{}^{\beta\gamma}{}_{,000} n_\alpha n_\beta \right.$$
$$\left. + 2\overset{(p)}{D}{}^{\alpha\beta}{}_{,000} \overset{(q)}{D}{}^{\alpha\beta}{}_{,000} \right\} n_\lambda \, d\Omega \tag{4.4}$$

to lowest multiple order. The integration of equation (4.4) proceeds in an analogous manner to the power flux calculation of Booth (1973). Equations (B.3.1) to (B.3.5) with equation (4.4) yield the linear momentum flux between the pth and qth radiators as

$${}_{\text{int}}\dot{P}_\lambda = \frac{G}{36c^6} \left\{ \left(\overset{(p)}{\bar{D}}{}^{\alpha\beta}{}_{,000} \overset{(q)}{D}{}^{\gamma\delta}{}_{,000} + \overset{(p)}{D}{}^{\alpha\beta}{}_{,000} \overset{(q)}{\bar{D}}{}^{\gamma\delta}{}_{,000} \right) \hat{n}_\alpha \hat{n}_\beta \hat{n}_\gamma \hat{n}_\delta \hat{n}_\lambda \right.$$
$$\cdot (\cos \rho_{pq} [-\rho_{pq}^{-1} + 105\rho_{pq}^{-3} - 945\rho_{pq}^{-5}] + \sin \rho_{pq} [15\rho_{pq}^{-2} - 420\rho_{pq}^{-4}$$
$$+ 945\rho_{pq}^{-6}]) + \left(\overset{(p)}{\bar{D}}{}^{\alpha\beta}{}_{,000} \overset{(q)}{D}{}^{\gamma\delta}{}_{,000} - \overset{(p)}{D}{}^{\alpha\beta}{}_{,000} \overset{(q)}{\bar{D}}{}^{\gamma\delta}{}_{,000} \right) {}^*\hat{n}_\alpha {}^*\hat{n}_\beta {}^*\hat{n}_\gamma {}^*\hat{n}_\delta {}^*\hat{n}_\lambda$$
$$\cdot (\cos {}^*\rho_{pq} [-{}^*\rho_{pq}^{-1} + 105{}^*\rho_{pq}^{-3} - 945{}^*\rho_{pq}^{-5}] + \sin {}^*\rho_{pq} [15{}^*\rho_{pq}^{-2}$$
$$- 420{}^*\rho_{pq}^{-4} + 945{}^*\rho_{pq}^{-6}]) + \left(\overset{(p)}{\bar{D}}{}^{\alpha\gamma}{}_{,000} \overset{(q)}{D}{}^{\beta\gamma}{}_{,000} + \overset{(p)}{D}{}^{\alpha\gamma}{}_{,000} \overset{(q)}{\bar{D}}{}^{\beta\gamma}{}_{,000} \right) \hat{n}_\alpha \hat{n}_\beta \hat{n}_\lambda$$

$$. \left(\cos \rho_{pq} \, [4\rho_{pq}^{-1} - 100\rho_{pq}^{-3} + 420\rho_{pq}^{-5}] + \sin \rho_{pq} \, [-28\rho_{pq}^{-2} + 240\rho_{pq}^{-4}\right.$$

$$\left. - 420\rho_{pq}^{-6}]\right) + \bigg(\overset{(p)}{\bar{D}}{}^{\alpha\gamma}{}_{,000}\overset{(q)}{D}{}^{\beta\gamma}{}_{,000} - \overset{(p)}{D}{}^{\alpha\gamma}{}_{,000}\overset{(q)}{\bar{D}}{}^{\beta\gamma}{}_{,000}\bigg) \, {}^*\hat{n}_\alpha {}^*\hat{n}_\beta {}^*\hat{n}_\lambda$$

$$. \left(\cos {}^*\rho_{pq} \, [4{}^*\rho_{pq}^{-1} - 100{}^*\rho_{pq}^{-3} + 420{}^*\rho_{pq}^{-5}] + \sin {}^*\rho_{pq} \, [-28{}^*\rho_{pq}^{-2}\right.$$

$$\left. + 240{}^*\rho_{pq}^{-4} - 420{}^*\rho_{pq}^{-6}]\right) + \bigg(\overset{(p)}{\bar{D}}{}^{\alpha\beta}{}_{,000}\overset{(q)}{D}{}^{\alpha\beta}{}_{,000} + \overset{(p)}{D}{}^{\alpha\beta}{}_{,000}\overset{(q)}{\bar{D}}{}^{\alpha\beta}{}_{,000}\bigg)\hat{n}_\lambda$$

$$. \left(\cos \rho_{pq} \, [-2\rho_{pq}^{-1} + 2\rho_{pq}^{-3} - 30\rho_{pq}^{-5}] + \sin \rho_{pq} \, [2\rho_{pq}^{-2} - 12\rho_{pq}^{-4} + 30\rho_{pq}^{-6}]\right)$$

$$+ \bigg(\overset{(p)}{\bar{D}}{}^{\alpha\beta}{}_{,000}\overset{(q)}{D}{}^{\alpha\beta}{}_{,000} - \overset{(p)}{D}{}^{\alpha\beta}{}_{,000}\overset{(q)}{\bar{D}}{}^{\alpha\beta}{}_{,000}\bigg)\hat{n}_\lambda$$

$$. \left(\cos {}^*\rho_{pq} \, [-2{}^*\rho_{pq}^{-1} + 2{}^*\rho_{pq}^{-3} - 30{}^*\rho_{pq}^{-5}] + \sin {}^*\rho_{pq} \, [2{}^*\rho_{pq}^{-2} - 12{}^*\rho_{pq}^{-4}\right.$$

$$\left. + 30{}^*\rho_{pq}^{-6}]\right) + \bigg(\overset{(p)}{\bar{D}}{}^{\alpha\lambda}{}_{,000}\overset{(q)}{D}{}^{\gamma\delta}{}_{,000} + \overset{(p)}{D}{}^{\alpha\lambda}{}_{,000}\overset{(q)}{\bar{D}}{}^{\gamma\delta}{}_{,000} + \overset{(p)}{\bar{D}}{}^{\gamma\delta}{}_{,000}\overset{(q)}{D}{}^{\alpha\lambda}{}_{,000}$$

$$+ \overset{(p)}{D}{}^{\gamma\delta}{}_{,000}\overset{(q)}{\bar{D}}{}^{\alpha\lambda}{}_{,000}\bigg) \, \hat{n}_\alpha\hat{n}_\gamma\hat{n}_\delta \, . \left(\cos \rho_{pq} \, [-20\rho_{pq}^{-3} + 210\rho_{pq}^{-5}]\right.$$

$$\left. + \sin \rho_{pq} \, [-2\rho_{pq}^{-2} + 90\rho_{pq}^{-4} - 210\rho_{pq}^{-6}]\right) + \bigg(\overset{(p)}{\bar{D}}{}^{\alpha\lambda}{}_{,000}\overset{(q)}{D}{}^{\gamma\delta}{}_{,000}$$

$$- \overset{(p)}{D}{}^{\alpha\lambda}{}_{,000}\overset{(q)}{\bar{D}}{}^{\gamma\delta}{}_{,000} + \overset{(p)}{\bar{D}}{}^{\gamma\delta}{}_{,000}\overset{(q)}{D}{}^{\alpha\lambda}{}_{,000} - \overset{(p)}{D}{}^{\gamma\delta}{}_{,000}\overset{(q)}{\bar{D}}{}^{\alpha\lambda}{}_{,000}\bigg) \, {}^*\hat{n}_\gamma {}^*\hat{n}_\delta {}^*\hat{n}_\alpha$$

$$. \left(\cos {}^*\rho_{pq} \, [-20{}^*\rho_{pq}^{-3} + 210{}^*\rho_{pq}^{-5}] + \sin {}^*\rho_{pq} \, [-2{}^*\rho_{pq}^{-2} + 90{}^*\rho_{pq}^{-4}\right.$$

$$\left. - 210{}^*\rho_{pq}^{-6}]\right) + \bigg(\overset{(p)}{\bar{D}}{}^{\alpha\lambda}{}_{,000}\overset{(q)}{D}{}^{\alpha\gamma}{}_{,000} + \overset{(p)}{D}{}^{\alpha\lambda}{}_{,000}\overset{(q)}{\bar{D}}{}^{\alpha\gamma}{}_{,000} + \overset{(p)}{\bar{D}}{}^{\alpha\gamma}{}_{,000}\overset{(p)}{D}{}^{\alpha\lambda}{}_{,000}\bigg)$$

$$+ \overset{(p)}{D}{}^{\alpha\gamma}{}_{,000}\overset{(q)}{\bar{D}}{}^{\alpha\lambda}{}_{,000})\hat{n}_\gamma \, . \left(\cos \rho_{pq} \, [16\rho_{pq}^{-3} - 60\rho_{pq}^{-5}] + \sin \rho_{pq} \, [4\rho_{pq}^{-2}\right.$$

$$\left. - 36\rho_{pq}^{-4} + 60\rho_{pq}^{-6}]\right) + \bigg(\overset{(p)}{\bar{D}}{}^{\alpha\lambda}{}_{,000}\overset{(q)}{D}{}^{\alpha\gamma}{}_{,000} - \overset{(p)}{D}{}^{\alpha\lambda}{}_{,000}\overset{(q)}{\bar{D}}{}^{\alpha\gamma}{}_{,000}$$

$$+ \overset{(p)}{\bar{D}}{}^{\alpha\gamma}{}_{,000}\overset{(q)}{D}{}^{\alpha\lambda}{}_{,000} - \overset{(p)}{D}{}^{\alpha\gamma}{}_{,000}\overset{(q)}{\bar{D}}{}^{\alpha\lambda}{}_{,000}\bigg){}^*\hat{n}_\gamma \, . \left(\cos {}^*\rho_{pq} \, [16{}^*\rho_{pq}^{-3} - 60{}^*\rho_{pq}^{-5}]\right.$$

$$\left. + \sin {}^*\rho_{pq} \, [4{}^*\rho_{pq}^{-2} - 36{}^*\rho_{pq}^{-4} + 60{}^*\rho_{pq}^{-6}]\right)\bigg\} \qquad (4.5)$$

where

$$\left.\begin{aligned}\overset{(i)}{D}{}^{\alpha\beta} &\equiv \overset{(i)}{D}{}^{\alpha\beta}(t - r/c)\\ \overset{(i)}{\bar{D}}{}^{\alpha\beta} &\equiv \overset{(i)}{D}{}^{\alpha\beta}(t - r/c + \pi/2\overset{(i)}{\omega})\end{aligned}\right\} \quad i = p, q \tag{4.6}$$

This then gives the total linear momentum loss rate for the galaxy as a whole as

$$_{\text{tot}}\dot{P}_\lambda = \tfrac{1}{2} \sum_{p,q=1}^{N} {}_{pq}\dot{P}_\lambda \tag{4.7}$$

where

$$_{pp}\dot{P}_\lambda = 0 \tag{4.8}$$

to quadrupole-quadrupole order.

5. *Discussion*

Whilst the present-day arena of gravitational radiation experiments is primarily concerned with the pure detection of energy-momentum carrying gravitational waves in order to confirm or deny Professor Weber's results (Weber, 1969, 1971), a time will surely come when technology will enable a more critical analysis of received signals. It is for this reason that the work in this paper and its predecessor was performed. It is hoped, at a later date, to complete this work with some numerical calculations based on current astrophysical data with the desire to present an idea of the orders of magnitude to be expected from specific galactic models.

Appendix A

In this appendix the spatial derivatives of the wave functions listed in equation (3.16) are given.

$$\begin{aligned}\bar{\psi}^{\alpha\beta}{}_{,\gamma} = &-\frac{2G}{c^4 r}\{\bar{d}^{\alpha\beta}{}_{,000}n_\gamma\} - \frac{2G}{c^4 r^2}\Big\{2\bar{d}^{\alpha\beta}{}_{,000}n_\gamma n_\delta \bar{L}^\delta - \bar{d}^{\alpha\beta}{}_{,000}\bar{L}^\gamma\\ &+ \bar{d}^{\alpha\beta}{}_{,00}n_\gamma + \frac{\bar{d}^{\alpha\beta}{}_{,0000}}{2}n_\gamma(n_\phi n_\pi \bar{L}^\phi \bar{L}^\pi - \bar{L}^\phi \bar{L}^\phi)\Big\}\end{aligned} \tag{A.1}$$

$$\begin{aligned}\bar{\psi}^{\alpha 0}{}_{,\gamma} = &-\frac{2G}{c^4 r}\{\bar{d}^{\alpha\beta}{}_{,000}n_\beta n_\gamma\} - \frac{2G}{c^4 r^2}\{3\bar{d}^{\alpha\beta}{}_{,000}n_\beta n_\gamma n_\delta \bar{L}^\delta\\ &- \bar{d}^{\alpha\beta}{}_{,000}(n_\gamma \bar{L}^\beta + n_\beta \bar{L}^\gamma) + 3\bar{d}^{\alpha\beta}{}_{,00}n_\beta n_\gamma - \bar{d}^{\alpha\gamma}{}_{,00}\\ &+ \tfrac{1}{2}\bar{d}^{\alpha\beta}{}_{,0000}n_\beta n_\gamma(n_\phi n_\pi \bar{L}^\phi \bar{L}^\pi - \bar{L}^\phi \bar{L}^\phi)\}\end{aligned} \tag{A.2}$$

$$\bar{\psi}^{00}{}_{,\gamma} = -\frac{2G}{c^4 r}\{\bar{d}^{\alpha\beta}{}_{,000}n_\alpha n_\beta n_\gamma\} - \frac{2G}{c^4 r^2}\{4\bar{d}^{\alpha\beta}{}_{,000}n_\alpha n_\beta n_\gamma n_\delta \bar{L}^\delta$$
$$- \bar{d}^{\alpha\beta}{}_{,000}(n_\gamma n_\alpha \bar{L}^\beta + n_\gamma n_\beta \bar{L}^\alpha + n_\alpha n_\beta \bar{L}^\gamma)$$
$$+ 6\bar{d}^{\alpha\beta}{}_{,00}n_\alpha n_\beta n_\gamma - 2\bar{d}^{\alpha\gamma}{}_{,00}n_\alpha - \bar{d}^{\delta\delta}{}_{,00}n_\gamma$$
$$+ \tfrac{1}{2}\bar{d}^{\alpha\beta}{}_{,0000}n_\alpha n_\beta n_\gamma(n_\phi n_\pi \bar{L}^\phi \bar{L}^\pi - \bar{L}^\phi \bar{L}^\phi)\} \tag{A.3}$$

The time derivatives follow immediately from equation (3.16).

Appendix B

B.1. Averaging Products of Components of a Unit Vector

The integrals to be considered are those of the type

$$\frac{1}{4\pi}\int_{4\pi} n_{\alpha_1} n_{\alpha_2} \ldots n_{\alpha_m} \exp i\,(\rho_\omega n_\omega)\, d\Omega \tag{B.1.1}$$

where

$$d\Omega = r^{-2}\, dS \tag{B.1.2}$$

where S is the unit sphere in 3-space with centre at the origin, m is any integer $\geqslant 0$ and (n_1, n_2, n_3) is the outward directed unit vector normal to S.

Write

$$\underline{\rho} = (\rho_1, \rho_2, \rho_3) = \rho(\hat{n}_1, \hat{n}_2, \hat{n}_3) \tag{B.1.3}$$

where

$$\underline{\hat{n}} \cdot \underline{\hat{n}} = 1 \tag{B.1.4}$$

The work considered in this appendix is an extension of that done previously by the present author (Booth, 1970) where the following identity was used†

$$\frac{1}{4\pi}\int_{4\pi} (d_\alpha n_\alpha)^{2p}\, d\Omega = \frac{(d_\alpha d_\alpha)^p}{(2p+1)} \tag{B.1.5}$$

where p is any integer $\geqslant 0$.

An analogous identity exists for integrals of the form (B.1.1) and the aim of the present work is, by use of this identity, to give a general expression which enables integrals of this form to be more readily evaluated when m is large and where the specific distribution of the normal components n_1, n_2, n_3 amongst the $n_{\alpha_1} \ldots n_{\alpha m}$ in the integrand of (B.1.1) is not known.

Clearly

$$\frac{1}{4\pi}\int_{4\pi} n_{\alpha_1} n_{\alpha_2} \ldots n_{\alpha_m} \exp i\,(\rho_\omega n_\omega)\, d\Omega = (-i)^m C^{(m)} \tag{B.1.6}$$

† This identity is quite easily proved by letting $\underline{d}$ be in the positive z-direction and using spherical polar coordinates.

where

$$C^{(m)} = \frac{\partial^m}{\partial\rho_{\alpha_1}\ldots\partial\rho_{\alpha_m}}\left\{\frac{1}{4\pi}\int_{4\pi}\exp i\,(\rho_\omega n_\omega)\,d\Omega\right\} \tag{B.1.7}$$

and

$$\frac{1}{4\pi}\int_{4\pi}\exp i\,(\rho_\omega n_\omega)\,d\Omega = \frac{\sin\rho}{\rho} \tag{B.1.8}$$

B.2. The General Expression for $C^{(m)}$

Instead of performing m differentiations to evaluate $C^{(m)}$ an alternative procedure is given by the following expression†, ‡

$$C^{(m)} = D^{(m)}\hat{n}_{\alpha_0}\hat{n}_{\alpha_1}\hat{n}_{\alpha_2}\ldots\hat{n}_{\alpha_m} + \sum_{r=1}^{\bar{m}}\left\{\frac{D^{(m-r)}}{\rho^r}\cdot\prod_{s=0}^{m-2r}\hat{n}_{(\alpha_s}\prod_{t=1}^{r}\delta_{\alpha_{m+1-2t},\,\alpha_{m+2-2t})}\right\} \tag{B.2.1}$$

where

$$D^{(m)} = \frac{\sin(\rho + m\pi/2)}{\rho} + \sum_{r=1}^{m}\left\{\frac{\sin(\rho + \overline{m+r}\pi/2)}{\rho^{r+1}.\,r!}\cdot\prod_{s=0}^{r-1}\left(\sum_{k=1}^{m}k - \sum_{p=0}^{s}p\right)\right\} \tag{B.2.2}$$

and §

$$\begin{aligned} \hat{n}_{\alpha_0} &= 1 \\ \bar{m} &= m/2 \qquad \text{if } m \text{ is even} \\ &= (m-1)/2 \qquad \text{if } m \text{ is odd.} \end{aligned} \tag{B.2.3}$$

B.3. List of Integrals

Using equation (B.1.6) or (B.2.1) the following integrals can be obtained

$m = 1$:

$$\frac{1}{4\pi}\int_{4\pi} n_\alpha \exp i\,(\rho_\omega n_\omega)\,d\Omega = -i\left(\frac{\cos\rho}{\rho} - \frac{\sin\rho}{\rho^2}\right)\hat{n}_\alpha \tag{B.3.1}$$

† The general expression for $C^{(m)}$ was derived by inspection for $m = 1, 2, 3$, etc., and the form deduced for the general case.

‡ The parentheses around the subscripts in (B.2.1) define the tensor symmetrisation procedure which involves all possible permutations of subscripts within the parentheses, e.g. $\delta_{(\alpha\beta)} = \frac{1}{2}\{\delta_{\alpha\beta} + \delta_{\beta\alpha}\}$. See footnote below.

§ $\hat{n}_{\alpha_0}$ is thus defined to enable the general expression to be written in the most elegant form. Note that before performing the symmetrisation procedure in (B.2.1) we put $\hat{n}_{\alpha_0} = 1$ and then symmetrise with respect to the remaining subscripts.

$m = 2$:

$$\frac{1}{4\pi}\int_{4\pi} n_\alpha n_\beta \exp i\,(\rho_\omega n_\omega)\,d\Omega$$

$$= -\left(\frac{\cos\rho}{\rho^2} - \frac{\sin\rho}{\rho^3}\right)\delta_{\alpha\beta} + \left(\frac{\sin\rho}{\rho} + \frac{3\cos\rho}{\rho^2} - \frac{3\sin\rho}{\rho^3}\right)\hat{n}_\alpha\hat{n}_\beta \tag{B.3.2}$$

$m = 3$:

$$\frac{1}{4\pi}\int_{4\pi} n_\alpha n_\beta n_\gamma \exp i\,(\rho_\omega n_\omega)\,d\Omega$$

$$= i\left(-\frac{\sin\rho}{\rho^2} - \frac{3\cos\rho}{\rho^3} + \frac{3\sin\rho}{\rho^4}\right)\hat{n}_{(\alpha}\delta_{\beta\gamma)}$$

$$+ i\left(-\frac{\cos\rho}{\rho} + \frac{6\sin\rho}{\rho^2} + \frac{15\cos\rho}{\rho^3} - \frac{15\sin\rho}{\rho^4}\right)\hat{n}_\alpha\hat{n}_\beta\hat{n}_\gamma \tag{B.3.3}$$

$m = 4$:

$$\frac{1}{4\pi}\int_{4\pi} n_\alpha n_\beta n_\gamma n_\delta \exp i\,(\rho_\omega n_\omega)\,d\Omega$$

$$= \left(-\frac{\sin\rho}{\rho^3} - \frac{3\cos\rho}{\rho^4} + \frac{3\sin\rho}{\rho^5}\right)\delta_{(\alpha\beta}\delta_{\gamma\delta)}$$

$$+ \left(-\frac{\cos\rho}{\rho^2} + \frac{6\sin\rho}{\rho^3} + \frac{15\cos\rho}{\rho^4} - \frac{15\sin\rho}{\rho^5}\right)\hat{n}_{(\alpha}\hat{n}_\beta\delta_{\gamma\delta)}$$

$$+ \left(\frac{\sin\rho}{\rho} + \frac{10\cos\rho}{\rho^2} - \frac{45\sin\rho}{\rho^3} - \frac{105\cos\rho}{\rho^4} + \frac{105\sin\rho}{\rho^5}\right)\hat{n}_\alpha\hat{n}_\beta\hat{n}_\gamma\hat{n}_\delta \tag{B.3.4}$$

$m = 5$:

$$\frac{1}{4\pi}\int_{4\pi} n_\alpha n_\beta n_\gamma n_\delta n_\pi \exp i\,(\rho_\omega n_\omega)\,d\Omega$$

$$= i\left(-\frac{\cos\rho}{\rho^3} + \frac{6\sin\rho}{\rho^4} + \frac{15\cos\rho}{\rho^5} - \frac{15\sin\rho}{\rho^6}\right)\hat{n}_{(\alpha}\delta_{\beta\gamma}\delta_{\delta\pi)}$$

$$+ i\left(\frac{\sin\rho}{\rho^2} + \frac{10\cos\rho}{\rho^3} - \frac{45\sin\rho}{\rho^4} - \frac{105\cos\rho}{\rho^5} + \frac{105\sin\rho}{\rho^6}\right)\hat{n}_{(\alpha}\hat{n}_\beta\hat{n}_\gamma\delta_{\delta\pi)}$$

$$+ i\left(\frac{\cos\rho}{\rho} - \frac{15\sin\rho}{\rho^2} - \frac{105\cos\rho}{\rho^3} + \frac{420\sin\rho}{\rho^4} + \frac{945\cos\rho}{\rho^5} - \frac{945\sin\rho}{\rho^6}\right)$$

$$\times\, \hat{n}_\alpha\hat{n}_\beta\hat{n}_\gamma\hat{n}_\delta\hat{n}_\pi \tag{B.3.5}$$

References

Bonnor, W. B. and Rotenberg, M. A. (1961). *Proceedings of the Royal Society* (*London*), **A265**, 109.

Bonnor, W. B. and Rotenberg, M. A. (1965). *Proceedings of the Royal Society* (*London*), **A289**, 247.

Booth, D. J. (1970). *SIAM Journal of Applied Mathematics*, **19**, 379.

Booth, D. J. (1973). *International Journal of Theoretical Physics*, Vol. 7, No. 5, p. 327.

Booth, D. J., Cooperstock, F. I. and Rumsey, J. (1972). *Nuovo Cimento*, **8B**, 90.

Cooperstock, F. I. and Booth, D. J. (1969a). *Nuovo Cimento*, **62B**, 163.

Cooperstock, F. I. and Booth, D. J. (1969b). *Physical Review*, **187**, 1796.

Landau, L. D. and Lifshitz, E. M. (1965). *The Classical Theory of Fields*, 2nd ed., revised. Addison-Wesley, U.S.A.

Møller, C. (1966). *Kongelige Danske Videnskabernes Selskabs Skrifter Mat.-Fys. Medd.*, **35**, (3).

Morgan, T. A. and Peres, A. (1963). *Physical Review*, **131**, 494.

Papapetrou, A. (1962). *Compte Rendu hebdomadaire des séances de l'Académie des Sciences*, **255**, 1578.

Peres, A. (1962). *Physical Review*, **128**, 2471.

Peters, P. C. (1964). *Physical Review*, **136**, 1224.

Weber, J. (1969). *Physical Review Letters*, **22**, 1320.

Weber, J. (1971). *International Seminar on Relativity and Gravitation, Israel 1969*, p. 309 et seq. Gordon and Breach, London.